A Graphic Presentation of Some Bitopological Spaces

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Abstract:

Given a bitopological space (X,τ_1,τ_2), where both (X,τ_1) and (X,τ_2) belong to a certain class of topological spaces , we will show that there exist a graph $G=(X,s_1,s_2)$ which will give a graphic presentation of the bitopological space (X,τ_1,τ_2).
Keywords: Graph; Bitopology; maps; Idempotent.

Preliminaries:

1-1. Definition: If X is a set, a map $s:X\to X$ is said to be an idempotent map if $s\circ s=s$.

1-2. Lemma: If $s:X\to X$ is any idempotent map, $C_s:P(X)\to P(X)$ defined by $C_s(A)=A\bigcup s(A)$ for any $A\in P(A)$, then C_s is a closure operation in the set X.

Proof: see [1].

1-3. Definition: If X is a non-empty set, $s:X\to X$ is an idempotent map. Let τ_s denotes the topology on X such that: $\overline{A}=A\bigcup s(A)$ for any $A\subset X$. We call τ_s the topology induced by the idempotent map s.

1-4. Definition: A space X is said to be $T_{1/2}$ space if and only if each one-point set is either open or closed in X.

The following theorem is 1.5.6 of [1].

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1-5. Theorem: If τ_s is the topology induced by an idempotent map $s: X \to X$, then the frontier of any one-point set is either empty or a one-point set.

1-6. Theorem: If τ_s is the topology induced by an idempotent map $s: X \to X$, then (X, τ_s) is a $T_{1/2}$ space.

Proof:

Let $x \in X$, since $s: X \to X$ is an idempotent map then either we have $s(x) = x$ or $s(x) = y$, $x \neq y$, and $s(y) = y$.

If $s(x) = x$, then $\overline{\{x\}} = \{x\} \cup s(\{x\}) = \{x\}$. So $\{x\}$ is a closed set.

If $s(x) = y$ where $x \neq y$, $s(y) = y$. Then $\overline{\{x\}} = \{x\} \cup s(\{x\}) = \{x, y\} = \{x\} \cup Fr\{x\}$, and so by 1-5 $Fr\{x\} = \{y\}$. Since $\{x\}^{\circ} = \{x\} \setminus Fr\{x\} = \{x\} \setminus \{y\} = \{x\}$, so $\{x\}$ is an open set.

1-7. Definition: A graph G is a triple (V, E, ψ), where V is a non-empty set called the set of vertices, E is a set disjoint from V called the set of edges, and ψ is a map from E into $V \times V$ called the incident map.

A graph $G = (V, E, \psi)$ is said to be directed graph if each edge is associated with an ordered pair of $V \times V$.

Now let $G = (V, E, \psi)$ be a directed graph, $\pi_i : V \times V \to V$ be the projection maps for $i = 1,2$, and let $d_i = \pi_i \circ \psi$ for $i = 1,2$. If we put $X = V \cup E$ and we let $s_i : X \to X$ be the map defined by:

$$s_i(x) = \begin{cases} x & if \quad x \in V \\ d_i(x) & if \quad x \in E \end{cases}$$

for $i = 1,2$. Then s_i is an idempotent map for, and s_1, s_2 satisfy the following composition property:

$s_2 \circ s_1 = s_1 \circ s_1 = s_1$,and $s_1 \circ s_2 = s_2 \circ s_2 = s_2$

So following [4] we can formalize the following equivalent definition of a directed graph.

1-8. Definition: A directed graph G is a triple (X, s_1, s_2), where X is a non-empty set and s_1, s_2 are two unary operations on X

satisfying the following composition property $s_2 \circ s_1 = s_1 \circ s_1 = s_1$, and $s_1 \circ s_2 = s_2 \circ s_2 = s_2$.

1-9. Definition: If X is a set and τ_1, τ_2 are two topologies on X, then the triple (X, τ_1, τ_2) is called a bitopological space.

Notice that if (X, τ_1, τ_2) is a bitopological space, $A \subset X$ then A is said to be τ_i-open if $A \in \tau_i$ for $i = 1,2$. And we say that (X, τ_1, τ_2) is a $T_{1/2}$ bitopological space if both $(X, \tau_1), (X, \tau_2)$ are $T_{1/2}$ spaces.

Graphic presentation:

In [4] Wald mar Korczynski gave a topological presentation of a graph and in [1] a topological presentation of a directed graph was given. In the following theorem, we will prove that the other way around works for a certain class of bitopological spaces.

2-1. Theorem: If (X, τ_1, τ_2) is a $T_{1/2}$ bitopological space, $Fr_{\tau_i}\{x\}$ is a one-element set or empty for all $x \in X$ and all i, and $\{x\}$ is τ_1-closed if and only if $\{x\}$ is τ_2-closed for all $x \in X$ .Then there exist a graph $G = (X, s_1, s_2)$ presenting the bitopological space (X, τ_1, τ_2).

Proof:

Let $s_i : X \to X$ defined by:

$$s_i(x) = \begin{cases} y & if \quad Fr_{\tau_i}\{x\} = \{y\} \quad , \quad y \neq x \\ x & if \quad Fr_{\tau_i}\{x\} = \{x\} \quad or \quad Fr_{\tau_i}\{x\} = \phi \end{cases}$$

Then s_i is an idempotent map for all i and $s_2 \circ s_1 = s_1$, $s_1 \circ s_2 = s_2$.

For let $x \in X$.

Case (1) If $\{x\}$ is closed. Then $Fr_{\tau_i}\{x\} = \{x\}$ or $Fr_{\tau_i}\{x\} = \phi$. So $(s_i \circ s_i)(x) = s_i(s_i(x)) = s_i(x)$.

Also $(s_2 \circ s_1)(x) = s_2(s_1(x)) = s_2(x) = x = s_1(x)$,

and $(s_1 \circ s_2)(x) = s_1(s_2(x)) = s_1(x) = x = s_2(x)$.

Case (2) If $\{x\}$ is open. Then $Fr_{\tau_1}\{x\} = \{y\}, y \neq x$,and $Fr_{\tau_2}\{x\} = \{z\}, \quad z \neq x$.

If $\{y\},\{z\}$ are τ_i-closed. Then since $Fr_{\tau_i}\{t\}=Fr_{\tau_i}[Fr_{\tau_i}(\{x\})]\subset Fr_{\tau_i}\{x\}=\{t\}$,where $t=y$ or z; that is $Fr_{\tau_1}\{y\}\subset\{y\}$, $Fr_{\tau_2}\{z\}\subset\{z\}$. So $Fr_{\tau_1}\{y\}=\{y\}$ or $Fr_{\tau_1}\{y\}=\phi$, and $Fr_{\tau_2}\{z\}=\{z\}$ or $Fr_{\tau_2}\{z\}=\phi$. Hence $(s_i\circ s_i)(x)=s_i(x)$ for $i=1,2$. Also $(s_2\circ s_1)(x)=s_1(x)$, and $(s_1\circ s_2)(x)=z=s_2(x)$.

The case $\{y\}$ is τ_1-open or $\{z\}$ is τ_2-open is impossible. Because without loss of generality if $\{y\}$ is τ_1-open and since $\{x\}$ is τ_1-open, then $\phi=Int_{\tau_1}(Fr_{\tau_1}\{x\})=Int_{\tau_1}\{y\}=\{y\}$ a contradiction.

Therefore s_1,s_2 satisfy the composition property:

$s_2\circ s_1=s_1\circ s_1=s_1$, and $s_1\circ s_2=s_2\circ s_2=s_2$ And hence the triple (X,s_1,s_2) is a graphic presentation of the bitopological space (X,τ_1,τ_2).

Example:

Let (X,τ_1,τ_2) be the bitopological space where $X=\{a, b, c, e_1, e_2\}$ and ,

$\tau_1=\{\phi, X, \{b\}, \{e_1\}, \{e_2\}, \{b, e_1\}, \{b, e_2\}, \{e_1, e_2\}, \{a, e_1\}, \{c, e_2\}, \{a, b, e_1\}, \{a, e_1, e_2\}, \{b, c, e_2\}, \{b, e_1, e_2\}, \{c, e_1, e_2\}, \{a, b, e_1, e_2\}, \{a, c, e_1, e_2\}, \{b, c, e_1, e_2\}\}$,and

$\tau_2=\{\phi, X, \{a\}, \{e_1\}, \{e_2\}, \{a, e_1\}, \{a, e_2\}, \{e_1, e_2\}, \{b, e_2\}, \{c, e_1\}, \{a, e_1, e_2\}, \{b, e_1, e_2\}, \{c, e_1, e_2\}, \{a, b, e_2\}, \{a, c, e_1\}, \{a, b, e_1, e_2\}, \{a, c, e_1, e_2\}, \{b, c, e_1, e_2\}\}$. Then

$Fr_{\tau_1}(\{a\})=\{a\}$, $Fr_{\tau_1}(\{b\})=\{b\}$, $Fr_{\tau_1}(\{c\})=\{c\}$, $Fr_{\tau_1}(\{e_1\})=\{a\}$, and $Fr_{\tau_1}(\{e_2\})=\{c\}$.

Let $s_1:X\to X$ be the map defined by

$$: s_1(x)=\begin{cases} x & if \quad x\neq e_1 \text{ and } x\neq e_2 \\ a & if \quad x=e_1 \\ c & if \quad x=e_2 \end{cases}$$

And $Fr_{\tau_2}(\{a\})=\{a\}$, $Fr_{\tau_2}(\{b\})=\{b\}$, $Fr_{\tau_2}(\{c\})=\{c\}$, $Fr_{\tau_2}(\{e_1\})=\{c\}$, and $Fr_{\tau_2}(\{e_2\})=\{b\}$.

Let $s_2 : X \to X$ be the map defined by :

$$s_2(x) = \begin{cases} x & if \quad x \neq e_1 \; and \; x \neq e_2 \\ c & if \quad x = e_1 \\ b & if \quad x = e_2 \end{cases}$$

Then figure 1.1 is the directed graph (X, s_1, s_2) which presents the bitopological space (X, τ_1, τ_2)

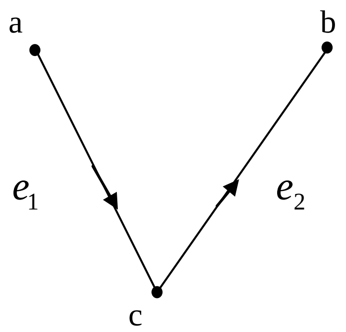

Figure 1.1

عرض بياني لبعض الفضاءات التبولوجية الثنائية

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المستخلص:

إذا أعطي أي فضاء توبولوجي ثنائي (X,τ_1,τ_2) حيث كل من (X,τ_1) ، (X,τ_2) ينتمي إلى صنف محدد من الفضاءات التوبولوجية. في هذه الورقة سوف نثبت بأنه يوجد بيان موجه $G=(X,s_1,s_2)$ يمثل عرض للفضاء التوبولوجي الثنائي (X,τ_1,τ_2).

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